

There is no square-complementary graph of girth 6

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Abstract

A graph is *square-complementary* (*squco*, for short) if its square and complement are isomorphic. We prove that there is no squco graph of girth 6, thus answering a question asked by Milanič et al. [Discrete Math., 2014, to appear], and leaving $g = 5$ as the only possible value of g for which the existence of a squco graph of girth g is unknown.

1 Introduction

Given two graphs G and H , we say that G is the *square* of H (and denote this by $G = H^2$) if their vertex sets coincide and two distinct vertices x, y are adjacent in G if and only if x, y are at distance at most two in H . Squares of graphs and their properties are well-studied in literature (see, e.g., Section 10.6 in the monograph [3]). A graph G is said to be *square-complementary* (*squco* for short) if its square is isomorphic to its complement. That is, $G^2 \cong \overline{G}$, or, equivalently, $G \cong \overline{G^2}$. The question of characterizing squco graphs was posed by Seymour Schuster at a conference in 1980 [10]. Since then, squco graphs were studied in the context of graph equations in terms of operators such as the line graph and complement (see [1, 2, 4–6, 9]). The entire set of solutions of some of these equations was found (see for example [1] and references quoted therein). However, the set of solutions of the equation $G^2 \cong \overline{G}$ remains unknown, despite several attempts to describe it (see for example [2, 5, 8]). The problem of determining all squco graphs was also posed as Open Problem No. 36 in Prisner's book [9].

Examples of squco graphs are K_1 , C_7 , and a cubic vertex-transitive bipartite squco graph on 12 vertices, known as the Franklin graph (see Fig. 1).

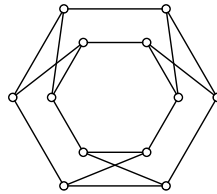


Figure 1: The Franklin graph.

The following two propositions, due to Baltić et al. [2] (and partially due to Capobianco and Kim [5]), summarize the results regarding the connectivity, radius, and diameter of squco graphs.

Proposition 1. *Every squco graph is connected and has no cut vertices.*

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Proposition 2. *If G is a nontrivial squco graph, then $\text{rad}(G) = 3$ and $3 \leq \text{diam}(G) \leq 4$. Moreover, if G is regular, then $\text{diam}(G) = 3$.*

It is not known whether a squco graph of diameter 4 exists. In the paper [8], several other questions regarding squco graphs were posed, and a summary of the known necessary conditions for squco graphs was given. Among them is the following result expressing a condition on the girth. (Recall that the *girth* of a graph G is the length of a shortest cycle in G , or ∞ if G is acyclic.)

Proposition 3. *If G is a nontrivial squco graph with girth at least 7, then G is the 7-cycle.*

This proposition leaves only 5 possible values for the girth g of a squco graph G , namely $g \in \{3, 4, 5, 6, 7\}$. The case $g = 7$ is completely characterized by Proposition 3. Baltić et al. [2] and Capobianco and Kim [5] asked whether there exists a squco graph of girth 3. An affirmative answer to this question was provided in [8] by a squco graph on 41 vertices with a triangle (namely, the circulant $C_{41}(\{4, 5, 8, 10\})$). As shown by the Franklin graph, there also exists a squco graph of girth 4. The questions regarding the existence of squco graphs of girth 5 or 6 were left as open questions in [8]. In this note, we answer one of them, by proving that there is no squco graph of girth 6. This leaves $g = 5$ as the only possible value of g for which the existence of a squco graph of girth g is unknown.

We briefly recall some useful definitions. Given two vertices u and v in a connected graph G , we denote by $d_G(u, v)$ the distance in G between u and v (that is, the number of edges on a shortest u - v path). For a positive integer i , we denote by $N_i(v, G)$ the set of all vertices u in G such that $d_G(u, v) = i$, and by $N_{\geq i}(v, G)$ the set of all vertices u in G such that $d_G(u, v) \geq i$. We use standard graph terminology [7].

2 The result

Theorem 1. *There is no squco graph of girth 6.*

Proof. Suppose for a contradiction that G is a squco graph of girth 6. First, we observe that if x is a vertex of G , then there are no edges in any of sets $N_i(x, G)$ for $i = 1, 2$ and no two distinct vertices in $N_1(x, G)$ have a common neighbor in $N_2(x, G)$. Let $k = \Delta(G)$ be the maximum degree of G , and let w be a vertex of degree k . Since the only squco graphs with maximum degree at most 2 are K_1 and C_7 [8], we have $k \geq 3$.

We consider two cases.

Case 1. w has a neighbor of degree at least three.

Let v be a neighbor of w of degree at least three, and let p and q be two neighbors of v other than w . If one of them, say p , is of degree at least 3, then p has at least two neighbors in $N_2(v, G)$ and thus $\Delta(\overline{G^2}) \geq |N_1(q, \overline{G^2})| \geq k + 1$, contrary to the fact that $\overline{G^2} \cong G$. Hence, both p and q are of degree 2. (Notice that Proposition 1 excludes the possibility of having degree 1 vertices). Let a and b be the unique neighbors of p and q in $N_2(v, G)$, respectively. The set $N_3(v, G)$ is nonempty, because radius of G is 3 by Proposition 2. Vertices a and b must be adjacent to all of vertices in $N_3(v, G)$, otherwise $\Delta(\overline{G^2}) \geq \max\{|N_1(p, \overline{G^2})|, |N_1(q, \overline{G^2})|\} \geq k + 1$, contrary to the fact that $\overline{G^2} \cong G$. To avoid a 4-cycle in G , we conclude that $|N_3(v, G)| = 1$. But now, the degree of v in $\overline{G^2}$ is 1, which implies that $\overline{G^2}$ has a cut vertex, contrary to the fact that G is squco and Proposition 1.

Case 2. All neighbors of w are of degree at most two.

In this case, all neighbors of w are of degree exactly two. In particular, $|N_2(w, G)| = |N_1(w, G)| = k \geq 3$. Now we will show that every vertex x from $N_2(w, G)$ is of degree at least

$|N_3(w, G)|$. Let $x \in N_2(w, G)$, and let y be the unique neighbor of x in $N_1(w, G)$. Vertex x has at least $|N_3(w, G)| - 1$ neighbors in $N_3(w, G)$, since otherwise $|N_1(y, \overline{G^2})| \geq k + 1$. This implies that any two vertices from $N_2(w, G)$ (the size of $N_2(w, G)$ is at least 3) have at least $|N_3(w, G)| - 2$ common neighbors in $N_3(w, G)$. This bounds $|N_3(w, G)| \leq 3$, otherwise we would have a 4-cycle.

Suppose $|N_3(w, G)| = 3$. To each of the three pairs of vertices in $N_3(w, G)$, associate, if possible, their common neighbor in $N_2(w, G)$. Because, each vertex in $N_2(w, G)$ is connected to at least two vertices in $N_3(w, G)$, it is surely associated with some pair. If $|N_2(w, G)| \geq 4$ then some two vertices from $N_2(w, G)$ are associated with the same pair and we get a 4-cycle, a contradiction. We thus have $|N_1(w, G)| = |N_2(w, G)| = k \leq 3$ and $|N_{\geq 4}(w, G)| = 0$ (otherwise we would have a vertex of degree at least $4 > k$ in $\overline{G^2}$). This implies that our graph has at most ten vertices. All squoco graphs with at most 11 vertices are known [8]; none of them has girth 6. Hence this is a contradiction with G having girth 6.

Suppose $|N_3(w, G)| = 2$. If $k \leq 4$, then we our graph has no more than 11 vertices, which is not possible. Hence $k \geq 5$. There must be at least $2k - 1$ vertices of degree two in G (all k vertices in $N_1(w, G)$; at most one of k vertices in $N_2(w, G)$ has both vertices from $N_3(w, G)$ for neighbors, otherwise we have a 4-cycle as before). In $\overline{G^2}$ at most $k + 3$ of them are of degree two, because every vertex in $N_1(w, G)$ will be connected to all but one vertex in $N_2(w, G)$ in $\overline{G^2}$, which is a contradiction, because $k \geq 5$.

The last possibility is that $|N_3(w, G)| = 1$, but then w would be of degree 1 in $\overline{G^2}$, again a contradiction. This completes the proof. \square

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